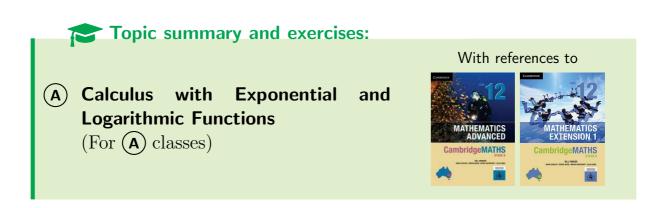


# MATHEMATICS ADVANCED YEAR 12 COURSE



Name:

Initial version by H. Lam, October 2013.

Last updated March 26, 2025, with major revision in March 2020 for Mathematics Advanced. Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under © CC BY 2.0.

### Symbols used



A Beware! Heed warning.



Mathematics Advanced content.



Mathematics Extension 1 content.



Literacy: note new word/phrase.

 $\mathbb{R}$  the set of real numbers

 $\forall$  for all

### Syllabus outcomes addressed

 ${\bf MA12\text{-}6}~$  applies appropriate differentiation methods to solve problems

MA12-7 applies the concepts and techniques of indefinite and definite integrals in the solution of problems

### Syllabus subtopics

MA-C2 Differential Calculus

MA-C3 Applications of Differentiation

 $\mathbf{MA\text{-}C4} \ \ \mathbf{Integral} \ \mathbf{Calculus}$ 

# Gentle reminder

- For a thorough understanding of the topic, every question in this handout is to be completed!
- Additional questions from CambridgeMATHS Year 12 Advanced or Cambridge-MATHS Year 12 Extension 1 will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

# Learning intentions & outcomes

Exp	onenti	al Functions: Differentiation
1	Co	ontent/learning intentions
	22.1	<b>2</b> Differentiation of exponential functions
	22.2	Establish and use the formula $\frac{d}{dx}(a^x) = (\log_e a) a^x$ Using graphing software or otherwise, sketch and explore the gradient function for a given exponential function, recognise it as another exponential function and hence determine the relationship between exponential functions and their derivatives
	22.3	Apply the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$ , $\frac{f(x)}{g(x)}$ and $f(g(x))$ where $f(x)$ and $g(x)$ are any of the functions covered in the scope of this syllabus $(ACMMM106)$ Also, establish using the chain rule: $\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$ and use the exponential laws to simplify an expression before differentiating.
	22.4	B Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems  Tangents and normals  Optimisation
Exp	onenti	al Functions: Integration
✓	Co	ontent/learning intentions
	22.5	Establish and use the formula $\int e^x dx = e^x + c$ and $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$
	22.6	(B) Calculate the area under a curve (ACMMM132)

1	Co	entent/learning intentions
	22.7	B Calculate areas between curves determined by any functions within the scope of this syllabus (ACMMM134)
	22.8	(B) Use the Trapezoidal rule to estimate areas under curves
Log	arithm	ic Functions: Differentiation
1	Co	entent/learning intentions
	22.9	Calculate the derivative of the natural logarithm function $\frac{d}{dx}(\ln x) = \frac{1}{x}$
	22.10	Establish and use the formula $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
	22.11	B Apply the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$ , $\frac{f(x)}{g(x)}$ and $f(g(x))$ where $f(x)$ and $g(x)$ are any of the functions covered in the scope of this syllabus (ACMMM106) Also, establish using the chain rule: $\frac{d}{dx} \left( \ln f(x) \right) = \frac{f'(x)}{f(x)}$ and use the logarithmic laws to simplify an expression before differentiating.
	22.12	<ul> <li>B Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems</li> <li>Tangents and normals</li> <li>Curve Sketching</li> <li>Optimisation</li> </ul>

# Logarithmic Functions: Integration

1	Content	/learning	intentions

- $\square$  22.14 **B** Establish and use the formulae  $\int a^x dx = \frac{a^x}{\ln a} + c$ 
  - **>** Use the change of base rule
- 22.15 (B) Calculate the area under a curve (ACMMM132)
- 22.16 B Calculate areas between curves determined by any functions within the scope of this syllabus (ACMMM134)
  - ➤ Functions whose primitive is a logarithmic function in this case, or if via a transformation to an alternative axis, the exponential function.
- 22.17 (B) Use the Trapezoidal rule to estimate areas under curves

# Curve Sketching with trigonometric functions Content/learning intentions $\mathbf{\mathcal{Z}}$ Examine and apply transformations to sketch functions of the form y=23.1 kf(a(x+b)) + c, where a, b, c and k are constants, in a variety of contexts, where f(x) is one of $\sin x$ , $\cos x$ or $\tan x$ , stating the domain and range when appropriate. ➤ Use technology or otherwise to examine the effect on the graphs of chang- $\bullet$ the amplitude (where appropriate) y = kf(x) $\bullet$ the period y = f(ax) $\bullet$ the phase f(x+b) $\bullet$ the vertical shift y = f(x) + c $\blacktriangleright$ $\square$ Use k, a, b and c to describe transformational shifts and sketch graphs. $\square$ Solve trigonometric equations involving functions of the form kf(a(x +23.2 b) + c, using technology or otherwise, within a specified domain

**2** Use trigonometric functions of the form kf(a(x+b)) + c to model and/or

solve practical problems involving periodic phenomena

23.3

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# Part I Calculus with Exponential Functions

# Section 1

# $oldsymbol{arepsilon}$ Differentiation involving the exponential function

# Learning Goal(s)

### **■** Knowledge

Transforming and differentiating the exponential function

### Skills

Differentiating the exponential function

### **V** Understanding

When to apply other differentiation techniques

### **☑** By the end of this section am I able to:

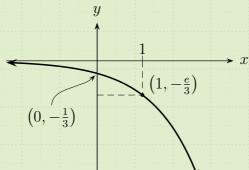
- 22.1 Differentiation of exponential functions
- 22.2 Establish and use the formula  $\frac{d}{dx}(a^x) = (\log_e a) a^x$  Apply the product, quotient and chain rules to differentiate functions of the form f(x)g(x),  $\frac{f(x)}{g(x)}$  and f(g(x)) where f(x) and g(x) are any of the functions covered in the scope of this syllabus
- 22.3 Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

# 1.1 Curve sketching without calculus

- Important note
- This section reviews work in Year 11 Topic 8 Exponentials and Logarithms. Only do as much as you have to.
- Consequently, only harder examples are given.

# Example 1

[Ex 5A/6A Q9] The graph is a dilation of  $y = e^x$ . Describe the dilation, and write down the equation of the curve:



**①** Timed exam practice 1 (Allow approximately 1 minute)

[2023 NBHS Adv Task 3] (1 mark) The function  $f(x) = e^{x+2}$  is transformed into the function  $g(x) = e^{x+4}$ . Which of the following describes the effect of this transformation?

- (A)A horizontal dilation
- (C) A vertical translation

(B) A vertical dilation (D)A reflection about the y axis

**=** Further exercises

- (A) Ex 5A (X1) Ex 6A
  - Only as much as you have to, to enjoy fluency in curve sketching

1.2 Differentiation

Example 2

[2017 2U HSC Q3] What is the derivative of  $e^{x^2}$ ? (A)  $x^2e^{x^2-1}$  (B)  $2xe^{2x}$  (C)  $2xe^{x^2}$ 

- (D)  $2e^{x^2}$

Example 3

[2018 2U HSC Q11] (2 marks) Differentiate  $\frac{e^x}{x+1}$ .



[Ex 5B/6B Q18] Define two functions  $\cosh x = \frac{e^x + e^{-x}}{2}$  and  $\sinh x = \frac{e^x - e^{-x}}{2}$ .

- (a) Show that  $\frac{d}{dx}(\cosh x) = \sinh x$  and  $\frac{d}{dx}(\sinh x) = \cosh x$ .
- (b) Find the second derivative of each function, and show that they both satisfy y'' = y.
- (c) Show that  $\cosh^2 x \sinh^2 x = 1$ .

# Further exercises

- (A) Ex 5B (x1) Ex 6B
  - Only as much as you have to, to enjoy fluency in differentiating via product/quotient/chain rules.

1.2.1 Change of base

Theorem 1

$$\frac{d}{dx}(a^x) = \frac{(\log_e a) a^x}{}$$

Proof



Let  $y = a^x$ . Take logarithm base e on both sides:

$$\therefore \quad \log_e y \quad = \quad x \log_e a$$

2. Exponentiate both sides, to base e:

$$y = \dots e^{x(\log_e a)}$$

3. Differentiate, noting the constants in the expression:

$$\frac{dy}{dx} = \log_e a e^{x(\log_e a)}$$

Replace  $e^{x(\log_e a)}$  with the expression it replaced:

$$\frac{dy}{dx} = \frac{(\log_e) a^x}{(\log_e) a^x}$$

**GeoGebra** 

 $\square$  Check that the derivative of  $y = a^x$  is simply a constant multiple of its corresponding y value.

# 1.3 Applications of differentiation

# 1.3.1 Tangents



[2014 2U HSC Q15] The line y = mx is a tangent to the curve  $y = e^{2x}$  at the point P.

i. Sketch the line and the curve on one diagram.

1

ii. Find the coordinates of P.

3

- iii. Find the value of m.
- **Answer:** i. Verify with technology. ii.  $P\left(\frac{1}{2}, e\right)$  iii. m = 2e

3

# 1.3.2 Curve sketching

# Theorem 2

The function  $e^x$  dominates over any algebraic power  $x^k, k \in \mathbb{Z}^+$  as  $x \to \pm \infty$ 

$$\lim_{x \to \infty} x^k e^{-x} = \dots \qquad \lim_{x \to -\infty} x^k e^x = \dots$$

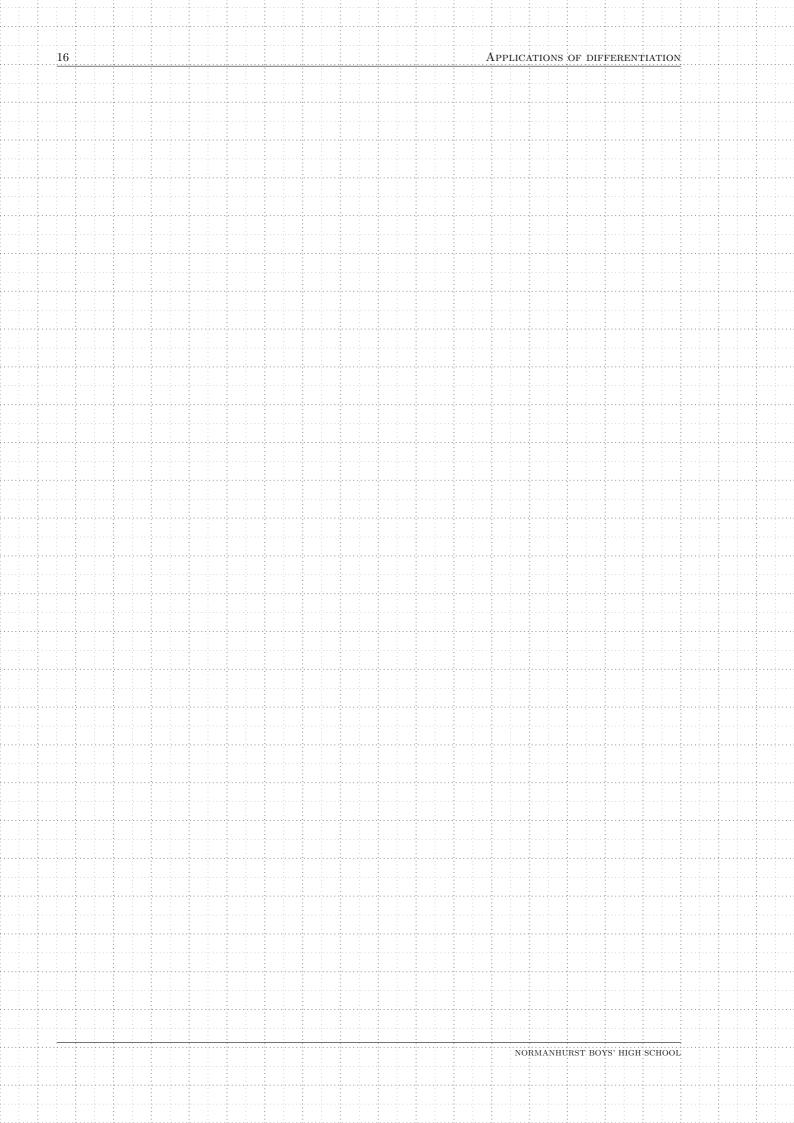
# Example 6

# [2024 SGS Adv Trial Q29] Let $f(x) = xe^{-x^2}$ .

(a) Find the coordinates of the stationary points of y = f(x). You do *not* need to check the nature of the stationary points.

Answer:  $\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}e}\right)$ 

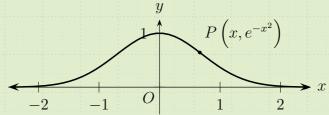
(b) Without using any further calculus, sketch the graph of y = f(x), showing stationary points, any intercepts and asymptotes.



# 1.3.3 Optimisation



[2018 Independent 2U Q16] In the diagram,  $P(x, e^{-x^2})$  is a variable point on the curve  $y = e^{-x^2}$ .



- i. Show that the function  $f(x) = e^{-x^2}$  is an even function.
- ii. Show that  $OP^2 = x^2 + e^{-2x^2}$ .
- iii. Find in simplest exact form, the minimum value of  $OP^2$  as the point P moves on the curve.

**Answer:**  $\frac{1}{2} (\ln 2 + 1)$ 

1

## 1.3.4 Modelling



[2020 Adv HSC Q21] Hot tea is poured into a cup. The temperature of tea can be modelled by  $T = 25 + 70(1.5)^{-0.4t}$ , where T is the temperature of the tea, in degrees Celsius, t minutes after it is poured.

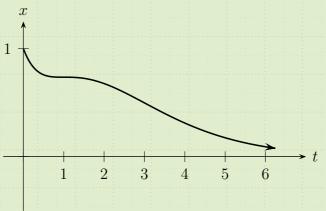
- (a) What is the temperature of the tea 4 minutes after it has been poured? 1
- (b) At what rate is the tea cooling 4 minutes after it has been poured? 2
- (c) How long after the tea is poured will it take for its temperature to reach 55°C?

  Answer: (a) 61.6°C (b) 5.9°C per minute (c) 5.22 minutes

### 1.3.5 Motion



[2023 Independent Adv Trial Q29] (5 marks) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line given by  $x = (t^2 + 1) e^{-t}$  and velocity v ms<sup>-1</sup>. The diagram shows the graph of x as a function of t.



- (a) Express v as a function of t and hence find when the particle is at rest. 2
- (b) Use the features of the graph to **explain** whether the velocity and acceleration of the particle are positive or negative over the period of time when it is speeding up.
- (c) **Describe** the limiting behaviour (speed and position) of the particle. 1

Important note

**A** Describe and explain questions require brief sentences!

## 1.3.6 Additional questions

- **1.** [1998 2U HSC Q6] The function  $f(x) = xe^{-2x} + 1$  has first derivative  $f'(x) = e^{-2x} 2xe^{-2x}$  and second derivative  $f''(x) = 4xe^{-2x} 4e^{-2x}$ .
  - i. Find the value of x for which y = f(x) has a stationary point.
  - ii. Find the values of x for which f(x) is increasing. 1
  - iii. Find the value of x for which y = f(x) has a point of inflection and determine where the graph of y = f(x) is concave upwards.
  - iv. Sketch the curve y = f(x) for  $-\frac{1}{2} \le x \le 4$ .
  - v. Describe the behaviour of the graph for very large values of x. 1

### Answers

**1.** (a)  $x = \frac{1}{2}$  (b)  $x < \frac{1}{2}$  (c) x = 1 (d) Sketch (e) As  $x \to \infty$ ,  $xe^{-2x} \to 0^+$ . Hence  $1 + xe^{-2x} \to 1^+$ . Also, as  $x \to -\infty$ ,  $xe^{-2x} \to -\infty$ . Hence  $1 + xe^{-2x} \to -\infty$ .

# Further exercises

- (A) Ex 5C (x1) Ex 6C
  - Only as much as you have to, to review applications of differentiation involving the exponential function.
- $\bigcirc$  Ex 7B

(x1) Ex 9B

• Q6, 8, 16

• Q5, 14

(A) Ex 7D

 $(x_1)$  Ex 9D

• Q6, 8, 14

• Q6, 11, **A** 13

# Section 2

# Integration involving the exponential function

# Learning Goal(s)

Various integration techniques involving the exponential func-

Finding primitives resulting in

**Vunderstanding** Why  $\int f'(x)e^{f(x)} dx = e^{f(x)} + C$ 

**☑** By the end of this section am I able to:

- Establish and use the formula  $\int e^x dx = e^x + c$  and  $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$
- 22.5 Calculate the area under a curve
- 22.6 Calculate areas between curves determined by any functions within the scope of this syllabus
- Use the Trapezoidal rule to estimate areas under curves

### Primitives resulting in $e^x$ 2.1

• Given  $\frac{d}{dx}(e^x) = e^x$ , then  $\int e^x dx = e^x + C$ 

# Laws/Results

- By the chain rule,  $\frac{d}{dx}(e^{f(x)}) = \dots f'(x)e^{f'(x)}$
- Hence,  $\int \dots f'(x)e^{f(x)} dx = e^{f(x)} + C.$

# Important note

**A** Think of "what needs to be differentiated to obtain the integrand?".

Chain rule will most likely be useful.

## 2.1.1 Indefinite integrals



# Example 10

Evaluate:

$$1. \qquad \int e^{-3x+2} \, dx.$$

$$4. \qquad \int \frac{e^{2x} + 1}{e^x} \, dx$$

$$2. \qquad \int \left(e^x + e^{-3x}\right) dx$$

$$5. \qquad \int \left(e^{3x} - 2\right)^2 dx$$

3. 
$$\int (e^x + 3) (e^x - 4) dx$$

$$6. \qquad \int 6^x \, dx$$

Answers

1. 
$$-\frac{4}{3}e^{-3x+2} + C$$
 2.  $e^x - \frac{1}{3}e^{-3x}$  3.  $\frac{1}{2}e^{2x} - e^x - 12x + C$  4.  $e^x - e^{-x} + C$  5.  $\frac{1}{6}e^{6x} - \frac{4}{3}e^{3x} + 4x + C$  6.  $\frac{1}{\ln 6}6^x + C$ 

# 2.1.2 Definite integrals



Evaluate 
$$\int_{2}^{3} e^{2-x} dx$$
.

Answer:  $\frac{e-1}{e}$ 

Evaluate 
$$\int_0^3 (2 + e^{-x}) dx$$
.

**Answer:**  $7 - \frac{1}{e^3}$ 

Evaluate 
$$\int_0^2 (e^{2x} + 3)^2 dx.$$

**Answer:**  $\frac{1}{4} (59 + 12e^4 + e^8)$ 

# 2.1.3 Harder primitives



Evaluate 
$$\int x^2 e^{x^3} dx$$

Answer: 
$$\frac{1}{3}e^{x^3} + C$$

Evaluate 
$$\int (3x - 6)e^{x^2 - 4x - 3} dx$$

**Answer:** 
$$\frac{3}{2}e^{x^2-4x-3} + C$$

Evaluate 
$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$
.

Answer: 
$$2e^{\sqrt{x}} + C$$

# Further exercises



(A) Ex 5D • Q1-4 last 2 columns

- $(x_1)$  Ex 6D
  - Q1 last column
  - Q2 last 2 columns
  - Q4

- Q5 last column
- Q7-9 last 2 columns
- Q11-18

• Q7-18

# 2.2 Applications of integration

### 2.2.1 Transformations

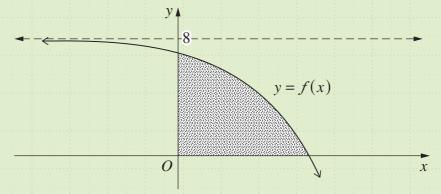
# **Theorem 3**

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

$$\int f'(x)a^{f(x)} dx = \frac{1}{\ln a}a^{f(x)} + C$$

# Example 17

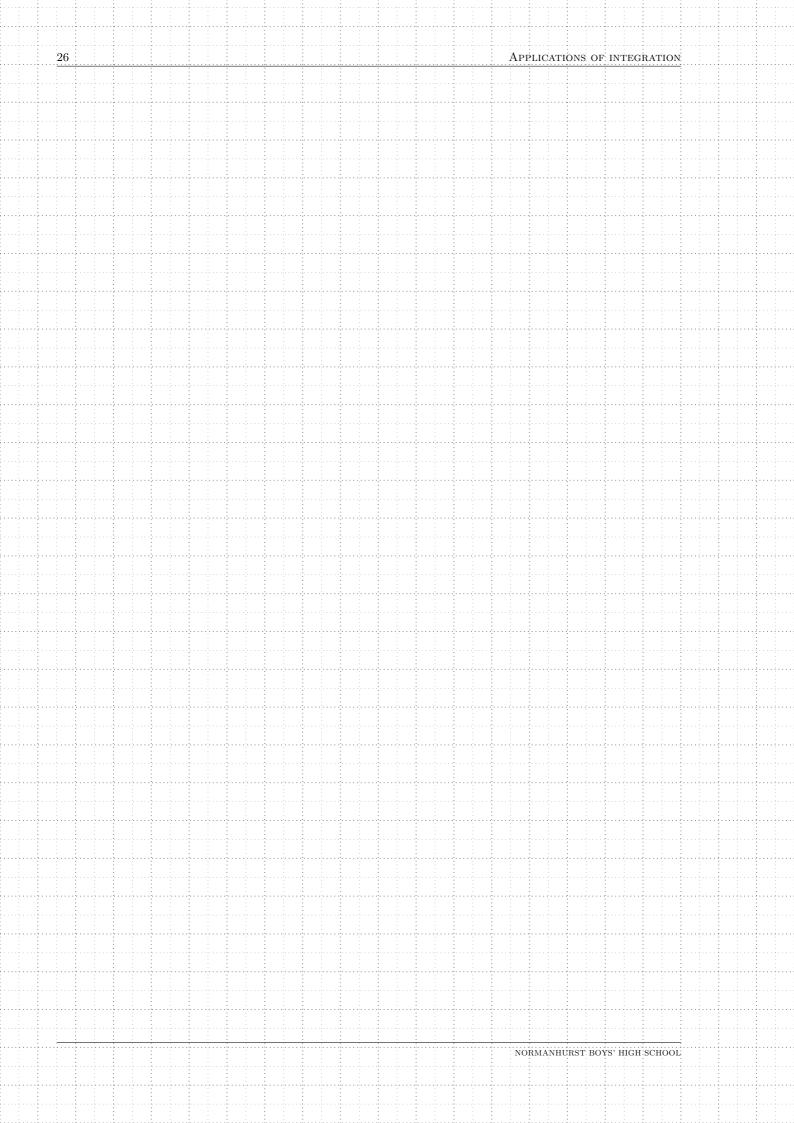
[2021 Adv HSC Q28] The region bounded by the graph of the function  $f(x) = 8 - 2^{-x}$  and the coordinate axes is shown.



- (a) Show that the exact area of the shaded region is given by  $24 \frac{7}{\ln 2}$
- (b) A new function g(x) is found by taking the graph of y = -f(-x) and translating it by 5 units to the right.

Sketch the graph of y = g(x) showing the x intercept and the asymptote.

(c) Hence, find the exact value of  $\int_{2}^{5} g(x) dx$  1



2.2.2 Simple areas

- Important note
- **A** Draw picture , if none is available!
- lacktriangle Check for x axis crossings.

# Example 18

Find the area between the curve  $y = e^x - e$  and the coordinate axes. Answer: 1

Example 19

[2013 CSSA Trial] Using the Trapezoidal Rule, find an approximation for

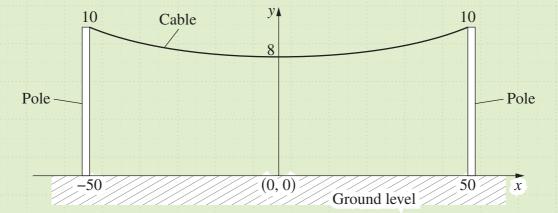
$$\int_0^2 e^{x^2} \, dx$$

with 3 function values.

**Answer:**  $\frac{1}{2} (1 + 2e + e^4)$ 



[2020 Adv HSC Sample Q38] A cable is freely suspended between two 10 m poles, as shown. The poles are 100 metres apart and the minimum height of the cable is 8 metres.



The height of the cable is given by  $y = c(e^{kx} + e^{-kx})$  where c and k are positive constants.

(a) Show that the value of c is 4.

1

(b) Use the result in part (a) to show that one value of k is  $\frac{\ln 2}{50}$ 

4

(c) Hence find the area between the poles, the cable and the ground.

3

Answer:  $\frac{600}{\ln 2} \approx 865 \,\mathrm{m}^2$ 

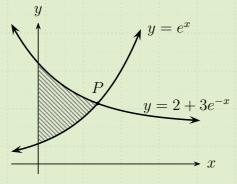
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### 2.2.3 Area between two curves

- Important note
- **A** Draw picture , if none is available!
- **A** Check for curves crossing over each other.

# Example 21

[2011 NSGHS 2U Trial] The diagram shows the graphs of  $y = e^x$  and  $y = 2 + 3e^{-x}$  intersecting at the point P.

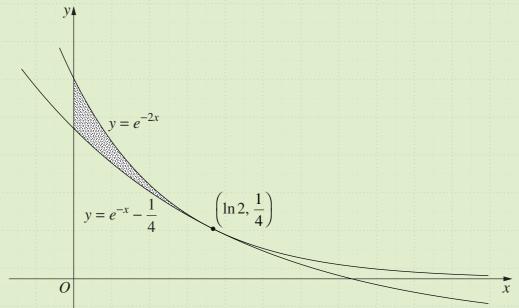


- (i) Show that the curves intersect when  $e^{2x} 2e^x 3 = 0$
- (ii) Hence show that the x coordinate of the point P is  $\log_e 3$ .
- (iii) Hence find the exact area of the shaded region.

3



[2023 Adv HSC Q32] The curves  $y = e^{-2x}$  and  $y = e^{-x} - \frac{1}{4}$  intersect at exactly one point as shown in the diagram. The point of intersection has coordinates  $\left(\ln 2, \frac{1}{4}\right)$ . (Do NOT prove this.)



(a) Show that the area bounded by the two curves and the y axis, as shaded in the diagram, is

 $\frac{1}{4}\ln 2 - \frac{1}{8}$ 

(b) Find the values of k such that the curves  $y = e^{-2x}$  and  $y = e^{-x} + k$  intersect at two points.

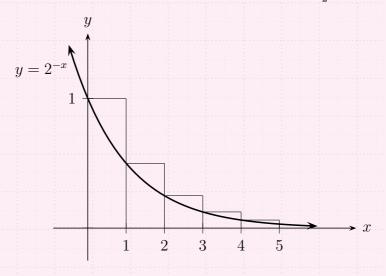
1

 $\mathbf{2}$ 

# **①** Timed exam practice 2 (Allow approximately 8 minutes)

[2022 Adv HSC Q29]

(a) The diagram shows the graph of  $y = 2^{-x}$ . Also shown on the diagram are the first 5 of an infinite number of rectangular strips of width 1 unit and height  $y = 2^{-x}$  for non-negative integer values of x. For example, the second rectangle shown has width 1 and height  $\frac{1}{2}$ .



The sum of the areas of the rectangles forms a geometric series.

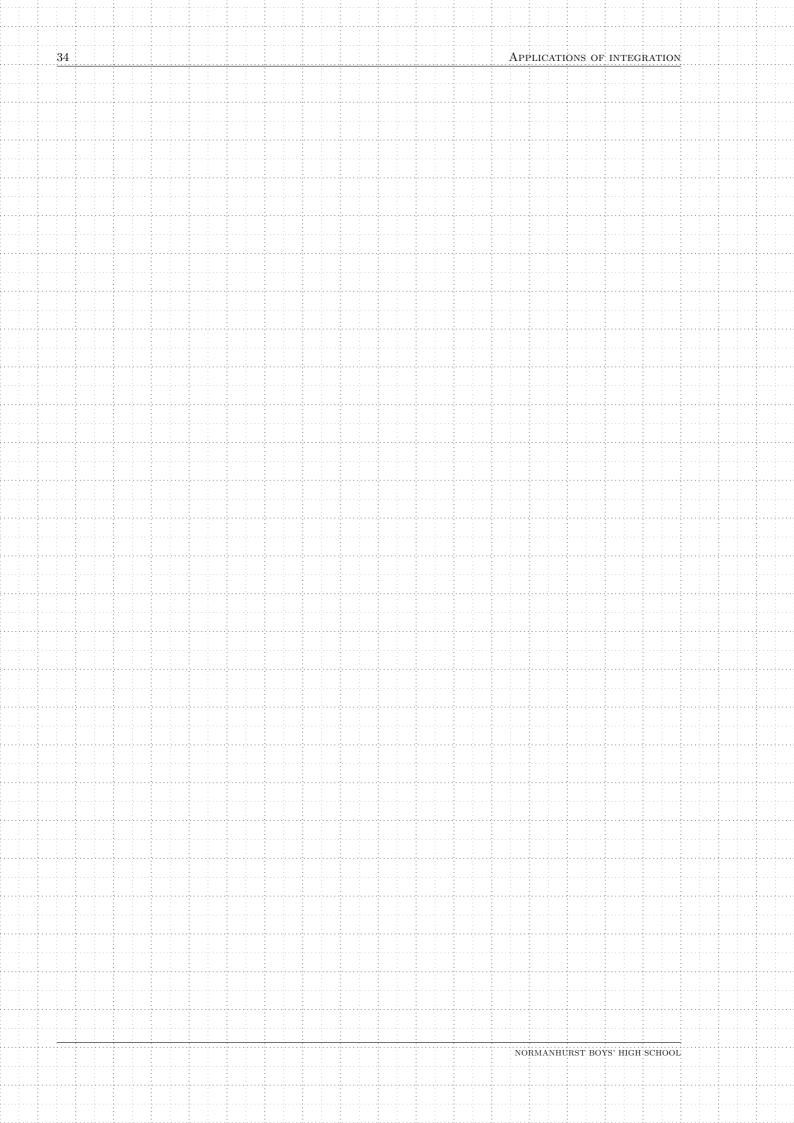
Show that the limiting sum of this series is 2.

(b) Show that 
$$\int_0^4 2^{-x} dx = \frac{15}{16 \ln 2}$$
.

(c) Use parts (a) and (b) to show that  $e^{15} < 2^{32}$ .

Marking criteria

- (a)  $\checkmark$  [1] Shows the limiting sum
- (b) \( \sqrt{1} \) Substitutes correctly into anti-derivative, or equivalent merit
  - ✓ [2] Provides correct solution
- (c)  $\checkmark$  [1] Sets up correct inequality, or equivalent merit
  - ✓ [2] Provides correct solution



### 2.2.4 Rates of change

### **①** Timed exam practice 3 (Allow approximately 7 minutes)

[2021 Adv HSC Q23] (4 marks) A population, P, which is initially 5 000, varies according to the formula

$$P = 5\,000b^{-\frac{t}{10}}$$

where b is a positive constant and t is time in years,  $t \geq 0$ .

The population is 1 250 after 20 years.

Find the value of t, correct to one decimal place, for which the instantaneous rate of decrease is 30 people per year.

Answer: t = 35.3Marking criteria

- $\checkmark$  [1] Provides correct derivative, or equivalent merit
- $\checkmark$  [2] Finds b and provides correct derivative, or equivalent merit
- $\checkmark$  [3] Equates -30 to a correct derivative, with b=2, or equivalent merit
- ✓ [4] Provides correct solution

### **2.2.5** Motion



[2017 2U HSC Q15] Two particles move along the x axis.

When t = 0, particle  $P_1$  is at the origin and moving with velocity 3.

For  $t \geq 0$ , particle  $P_1$  has acceleration given by  $a_1 = 6t + e^{-t}$ .

(i) Show that the velocity of particle 
$$P_1$$
 is given by  $v_1 = 3t^2 + 4 - e^{-t}$ . When  $t = 0$ , particle  $P_2$  is also at the origin.

For  $t \ge 0$ , particle  $P_2$  has velocity given by  $v_2 = 6t + 1 - e^{-t}$ .

(iii) Show that the particles do not meet for 
$$t > 0$$
.

## **①** Timed exam practice 4 (Allow approximately 3 minutes)

[2019 2U HSC Q14] (2 marks) A particle is moving along a straight line. The particle is initially at rest. The acceleration of the particle at time t seconds is given by  $a = e^{2t} - 4$ , where  $t \ge 0$ .

Find an expression, in terms of t, for the velocity of the particle.

**Answer:**  $v = \frac{1}{2}e^{2t} - 4t - \frac{1}{2}$ 

#### Marking criteria

- $\checkmark$  [1] Attempts to integrate, or equivalent merit
- ✓ [2] Provides correct solution

## **Further exercises**

- (A) Ex 5E
  - All questions, except Q1-2
- $\bigcirc$  Ex 7B
  - Q8, 16
- $\bigcirc$  Ex 7C
  - Q8, 9: parts (c), (d)
  - Q16
- $\bigcirc$  Ex 7E
  - Q1(e), 8, 9

- (x1) Ex 6E
  - Q2-17
- $(x_1)$  Ex 9B
  - Q5, 14
- (x1) Ex 9C
  - Q4(a), (b): part ii
  - Q12
- $(x_1)$  Ex 9F
  - Q5, 6

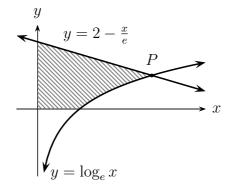
#### 2.2.6 Further questions

- 1. [Legacy Ex 13C Q13]
  - (a) Differentiate  $y = x^2 e^{-x^2}$ .
  - (b) Hence show that

$$\int x^3 e^{-x^2} dx = -\frac{e^{-x^2}}{2} (x^2 + 1) + C$$

and calculate  $\int_{1}^{2} x^{3}e^{-x^{2}} dx$ .

2. [2011 Independent 2U Trial] (xi) The shaded region below represents the area bounded by the x and y axes and the curves  $y = 2 - \frac{x}{e}$  and  $y = \log_e x$ .



- (i) Show by substitution that the curves  $y = 2 \frac{x}{e}$  and  $y = \log_e x$  intersect at the point P(e, 1).
- (ii) Hence find the exact area of the shaded region. Answer:  $\frac{3e}{2} 1$  4
- 3. [2020 Adv HSC Sample Q30] (3 marks) The population, P, of rabbits on an island is given by P(t), where t is the time in years after the rabbits were introduced. The rabbit population changes at a rate modelled by the function  $\frac{dP}{dt} = 30e^{1.25t}$ .

Calculate the increase in the number of rabbits at the end of the first 10 years. Give your answer correct to two significant figures.

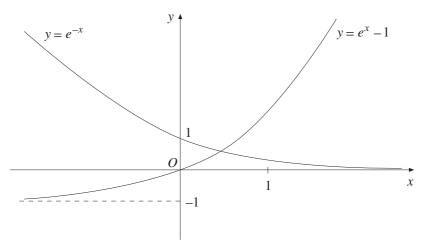
Answer: 6 400 000

 $\mathbf{2}$ 

3

#### 4. [1996 2U HSC Q5]

- (b) Solve the equation  $u^2 u 1 = 0$ , correct to three decimal places.
- (c) The diagram shows the graphs of  $y = e^x 1$  and  $y = e^{-x}$ .



- i. Find the area between the curves from x = 1 to x = 2. Leave your answer in terms of e.
- ii. Show that the curves intersect when 2

$$e^{2x} - e^x - 1 = 0$$

iii. Use the results of part (b) with  $u = e^x$  to show that the x coordinate of the point of intersection of the curves is approximately 0.481.

x x

# Part II Calculus with Logarithmic Functions

## Section 3

# Differentiation involving the logarithmic function



## Learning Goal(s)

Transforming and differentiating the logarithmic function

Differentiating the logarithmic

Why  $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ , and when to apply other differentiation

**☑** By the end of this section am I able to:

- Calculate the derivative of the natural logarithm function  $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- 22.10 Establish and use the formula  $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- Apply the product, quotient and chain rules to differentiate functions of the form f(x)g(x),  $\frac{f(x)}{g(x)}$ and f(g(x)) where f(x) and g(x) are any of the functions covered in the scope of this syllabus
- 22.12 Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

## 3.1 **2** Curve sketching without calculus and equations



Solve  $3e^{2x} - 11e^x - 4 = 0$ .

Answer: ln 4

Solve  $\ln x - \frac{9}{\ln x} = 0$ 

**Answer:**  $e^3$  or  $e^{-3}$ 

**Example 26** Sketch  $y = \log_e(5 - x)$  by transforming the graph of  $y = \log_e x$ .



#### Timed exam practice 5 (Allow approximately 1 minute)

[2024 NBHS Adv Task 3] The function  $f(x) = \log_e x$  is transformed to  $g(x) = \log_{10}(2x)$  by using only two function transformations.

Which of the following transformations, applied in the order given, will transform f(x) into g(x)?

- (A)**1.** Apply a vertical dilation factor of  $\log_{10} e$ 
  - **2.** Apply a horizontal dilation factor of 2
- **1.** Apply a vertical dilation factor of  $\frac{1}{\log_{10} e}$ (B)
  - **2.** Apply a horizontal dilation factor of 2
- (C) **1.** Apply a vertical dilation factor of  $\log_{10} e$ 
  - 2. Apply a vertical translation of  $(\log_{10} 2)$  units in the upwards direction.
- **1.** Apply a vertical dilation factor of  $\frac{1}{\log_{10} e}$ (D)
  - 2. Apply a vertical translation of  $(\log_{10} 2)$  units in the upwards direction.



#### 3.1.1 Additional questions

- 1. [Ex 6F Q17] Use and describe:
  - (a) a dilation of  $y = \log_e x$ , to sketch  $y = \log_e 2x$ .
  - (b) a subsequent translation to sketch  $y = \log_e 2(x 1)$ .
  - (c) a subsequent dilation to sketch  $y = \frac{1}{2} \log_e 2(x-1)$ .
  - (d) a subsequent translation to sketch  $y = \frac{1}{2} \log_e 2(x-1) 2$ .

**=** Further exercises



(x1) Ex 6F

• Q5-16

Only as many as required to be fluent at equation solving/curve sketching.

## 3.2 Differentiation

- 3.2.1 Derivation of result
  - **Steps**
  - 1. If  $y = \log_e x$ .
  - 2. Change subject to x:  $x = e^y$
  - 3. Differentiate x with respect to y:  $\dots \frac{dx}{dy} = e^y$ ....
  - 4. Use the fact that  $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ :

**♣** Laws/Results

If  $y = \log_e x$  (or  $y = \ln x$ ), the derivative is

$$y = \frac{1}{x}$$





**A** The <u>chain</u> <u>rule</u> is almost always required.

#### 3.2.2 Standard questions



Example 27

1. 
$$\frac{d}{dx}\left(x + \log_e x\right)$$

$$2. \quad \frac{d}{dx} \left( \log_e \left( x^2 + 1 \right) \right)$$

1. 
$$\frac{d}{dx}(x + \log_e x)$$
 2.  $\frac{d}{dx}(\log_e (x^2 + 1))$  3.  $\frac{d}{dx}(\log_e (\log_e x))$ 

Answers

1. 
$$1 + \frac{1}{x}$$
 2.  $\frac{2x}{x^2+1}$  3.  $\frac{1}{x \log_e x}$ 



**♣** Laws/Results

**Shortcut:** If  $y = \log_e(f(x))$  or  $y = \ln(f(x))$ , the derivative is

$$y = \frac{f'(x)}{f(x)}$$

where f'(x) is the <u>chain</u> rule residue .

## 3.2.3 Miscellaneous questions

Example 28
Differentiate 
$$y = (x + \ln x)^4$$

**Answer:** 
$$4(1+\frac{1}{x})(x+\ln x)^3$$

**Answer:**  $\frac{3(2x+3)-2(3x+1)\ln(3x+1)}{(2x+3)^2(3x+1)}$ 

Example 29
Differentiate  $y = \frac{\ln(3x+1)}{2x+3}$ .

## 3.2.4 Simplification via log laws



Evaluate 
$$\frac{d}{dx} \left( \log_e \left( \frac{1+x^2}{1-x^2} \right) \right)$$

**Answer:** 
$$2x \left( \frac{1}{1+x^2} + \frac{1}{1-x^2} \right)$$

Evaluate 
$$\frac{d}{dx} \left( \log_e \sqrt{\frac{3x+4}{5x^2+1}} \right)$$

**Answer:** 
$$\frac{1}{2} \left( \frac{3}{3x+4} - \frac{10x}{5x^2+1} \right)$$

Example 32 Differentiate 
$$y = \log_e [(x^3 + 3)(x^2 + 3x + 1)].$$

**Answer:** 
$$\frac{3x^2}{x^3+3} + \frac{2x+3}{x^2+3x+1}$$

3.2.5 Change of base

Theorem 4

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$$

Proof



1. Let  $y = \log_a x$ . Exponentiate both sides, to base a:

$$a^y = x$$

**2.** Take logarithm base e on both sides:

$$\log_e a^y = \log_e x$$

3. Use a logarithm law to simply the y expression, and then divide:

$$y = \frac{1}{\log_e a} \log_e x$$

4. Differentiate, noting the constants in the expression:

$$\frac{dy}{dx} = \frac{1}{\log_e a} \times \frac{1}{x}$$

**5.** Fully simplify:

$$\frac{dy}{dx} = \frac{1}{x \log_e a}$$

Evaluate 
$$\frac{d}{dx} \left( \log_3(2x^3 + 3x) \right)$$

Answer: 
$$\frac{1}{\ln 3} \left( \frac{6x^2+3}{2x^3+3x} \right)$$

## Example 34

[2020 Adv HSC Sample Q10] Given the function  $y = \log_7(x^x)$ , which expression

is equal to  $\frac{dy}{dx}$ ?
(A)  $\frac{1}{x \ln 7}$ 

(A) 
$$\frac{1}{x \ln 7}$$

(B) 
$$\frac{1}{\ln 7} \times \log_7(x^{x-1})$$

(C) 
$$\frac{1}{x^2 \ln 7}$$

(C) 
$$\frac{1}{x^2 \ln 7}$$
  
(D)  $\log_7 x + \frac{1}{\ln 7}$ 

- (A) Ex 5G Q1-4 last 2 columns

- (x1) Ex 6G
  - Q3-4
  - Q5-12, 14 last 2 columns

• Q5-14, 17-18

• Q13

DIFFERENTIATION 51

#### 3.2.6 Supplementary Exercises

Differentiate the following with respect to x.

1. 
$$y = 4 \log_e x$$

$$2. y = \frac{\log_e x}{14}$$

3. 
$$y = \log_e(3x - 4)$$

4. 
$$y = \log_e(5x - 6)$$

5. 
$$y = \log_e(x^2)$$

**6.** 
$$y = \log_e (ax^2 + b)$$

7. 
$$y = \log_e (1 - x^2)$$

8. 
$$y = \log_e (-x^2 + 6x)$$

9. 
$$y = \log_e (2x^3 + 3x)$$

10. 
$$y = \log_e (2x^4 - 3x^2 + 2x + 1)$$

11. 
$$y = x \log_e x$$

**12.** 
$$y = x^2 \log_e x$$

13. 
$$y = x^2 \log_e(4x + 3)$$

**14.** 
$$y = (2x+5)^2 \log_a(2x+5)$$

15. 
$$y = \log_a (x^2 + x)$$

$$16. \quad y = \frac{\log_e x}{x}$$

$$17. \quad y = \frac{x^2}{\log_e x}$$

18. 
$$y = \frac{x^2 - 1}{\log_e x}$$

**19.** 
$$y = \log_e x^{100}$$

**20.** 
$$y = \log_e (x^2 + 4x + 5)^3$$

**21.** 
$$y = 6 \log_e \sqrt[3]{x}$$

**22.** 
$$y = \log_e \sqrt{1 + x^2}$$

**23.** 
$$y = \log_e \left( \frac{x^2}{1 + x^2} \right)$$

$$24. \quad y = \log_e \left( \frac{1+x}{1-x} \right)$$

**25.** 
$$y = \log_e \left( \frac{2x+3}{3x-4} \right)$$

**26.** 
$$y = \log_e \sqrt[4]{\frac{1+x^2}{1-x^2}}$$

**27.** 
$$y = \log_e \sqrt{\frac{x^4 - 1}{x^4 + 1}}$$

**28.** 
$$y = \log_e \left[ (x^2 + 2)^2 (x^3 + x - 1) \right]$$

**29.** 
$$y = \log_e \left[ (4x+2)^4 (8x-3)^6 \right]$$

**30.** 
$$y = \log_e (x\sqrt{2x+1})$$

**31.** 
$$y = \log_e \frac{x}{\sqrt{2x+1}}$$

**32.** 
$$y = (x^2 + 1)\log_e(2x + 1)$$

**33.** 
$$y = (ax + b) \log_e(ax)$$

**34.** 
$$y = \log_e x^3 + \log_e^3 x$$

**35.** 
$$y = x^{\log_e 3}$$

**36.** 
$$y = \log_e^4(ax)$$

**37.** 
$$y = \log_e^2(2x+3)$$

**38.** 
$$y = x \log_e \sqrt{x - 1}$$

**39.** 
$$y = \log_{a} \left( x^{2} \sqrt{3x - 2} \right)$$

**40.** 
$$y = \sqrt{4 + \log_e x}$$

**41.** 
$$y = \log_e \left( x + \sqrt{1 + x^2} \right)$$

#### Answers

1.  $\frac{4}{x}$  2.  $\frac{1}{14x}$  3.  $\frac{3}{3x-4}$  4.  $\frac{5}{5x-6}$  5.  $\frac{2}{x}$  6.  $\frac{2ax}{ax^2+b}$  7.  $\frac{2x}{x^2-1}$  8.  $\frac{-2x}{x^2+6x}$  9.  $\frac{6x^2+3}{2x^3+3x}$  10.  $\frac{8x^3-6x+2}{2x^4-3x^2+2x+1}$  11.  $1 + \log_e x$  12.  $x + 2x \log_e x$  13.  $\frac{4x^2}{4x+3} + 2x \log_e (4x+3)$  14.  $2(2x+5)(1+2\ln(2x+5))$  15.  $\frac{2x+1}{x^2+x}$  16.  $\frac{1-\log_e x}{x^2}$  17.  $\frac{x(2\log_e x-1)}{(\log_e x)^2}$  18.  $\frac{2x^2\log_e x-x^2+x}{x(\log_e x)^2}$  19.  $\frac{100}{x}$ 

**20.** 
$$\frac{3(2x+4)}{x^2+4x+5}$$
 **21.**  $\frac{2}{x}$  **22.**  $\frac{x}{x^2+1}$  **23.**  $\frac{2}{x} - \frac{2x}{1+x^2}$  **24.**  $\frac{1}{1+x} + \frac{1}{1-x}$  **25.**  $\frac{2}{2x+3} - \frac{3}{3x-4}$  **26.**  $\frac{1}{2} \left( \frac{x}{1+x^2} + \frac{x}{1-x^2} \right)$ 

$$\mathbf{27.} \ 2 \left( \frac{x^3}{x^4 - 1} - \frac{x^3}{x^4 + 1} \right) \ \mathbf{28.} \ \frac{4x}{x^2 + 2} + \frac{3x^2 + 1}{x^3 + x - 1} \ \mathbf{29.} \ \frac{8}{2x + 1} + \frac{48}{8x - 3} \ \mathbf{30.} \ \frac{3x + 1}{2x^2 + x} \ \mathbf{31.} \ \frac{1}{x} - \frac{1}{2x + 1} \ \mathbf{32.} \ \frac{2(x^2 + 1)}{2x + 1} + 2x \log_e(2x + 1)$$

33. 
$$\frac{1}{x}(ax+b)+a\log_e(ax)$$
 34.  $\frac{3}{x}\left(1+(\log_e x)^2\right)$  35.  $(\log_e 3)x^{\log_e 3-1}$  36.  $\frac{4a}{x}\left(\log_e(ax)\right)^3$  37.  $\frac{4\log_e(2x+3)}{2x+3}$  38.  $\frac{1}{2}\left(\frac{x}{x-1}+\log_e(x-1)\right)$ 

**39.**  $\frac{15x-8}{2(3x^2-2x)}$  **40.**  $\frac{1}{2x\sqrt{4+\log_e x}}$  **41.**  $\frac{1}{\sqrt{1+x^2}}$ 

## 3.3 Applications of differentiation

## 3.3.1 Tangents and normals

Example 35

Find the equation of the tangent to the curve  $y = \ln(3x - 1)$  at the point  $(2, \ln 5)$ Answer:  $y = \frac{3}{5}x - \frac{6}{5} + \ln 5$ 

## Example 36

- (a) Show that the tangent of the curve to  $y = \log_e x$  at T(e, 1) has equation x = ey.
- (b) Find the equation of the normal to  $y = \log_e x$  at T(e, 1).
- (c) Sketch the curve, the tangent and the normal, and find the area of the triangle formed by the y axis and the tangent and normal at T.

**Answer:** (a) Show (b)  $y = -ex + e^2 + 1$  (c)  $\frac{1}{2}e(e^2 + 1)$ 

#### 3.3.2 Curve sketching

#### Theorem 5

The function x dominates over any logarithmic function  $y = \log x$ , as  $x \to \infty$  or  $x \to 0^+$ .

$$\lim_{x \to \infty} \frac{\log x}{x} = \dots \qquad \lim_{x \to 0^+} x \log x = \dots$$

## **⊙** Timed exam practice 6 (Allow approximately 8 minutes)

[2004 2U HSC Q9] Consider the function  $f(x) = \frac{\log_e x}{x}$  for x > 0.

- i. Show that the graph of y = f(x) has a stationary point at x = e.
- ii. By considering the gradient on either side of x = e, show that the stationary point is a maximum.
- iii. Use the fact that the maximum value of f(x) occurs at x = e to deduce that  $e^x > x^e$  for all x > 0.

## Example 37

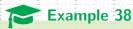
- Consider the curve  $f(x) = \log_e (4 x^2)$ . (a) Find the value(s) of x for which  $4 x^2 > 0$ .
- Hence write the domain of  $f(x) = \log_e(4 x^2)$ . (b)
- Find the x intercept(s) of  $f(x) = \log_e (4 x^2)$ . (c)
- (d) Find the first and second derivatives of f(x).
- (e) Hence show that f(x) has one stationary point and determine its nature.
- Sketch the curve  $f(x) = \log_e (4 x^2)$ . (f)

**Answer:** (a) 
$$-2 < x < 2$$
 (b) As above (c)  $x = \pm \sqrt{3}$  (d)  $f'(x) = -\frac{2x}{4-x^2}$ ,  $f''(x) = \frac{-2(4+x^2)}{(4-x^2)^2}$ . (e) max:  $(0, \ln 4)$ 

2

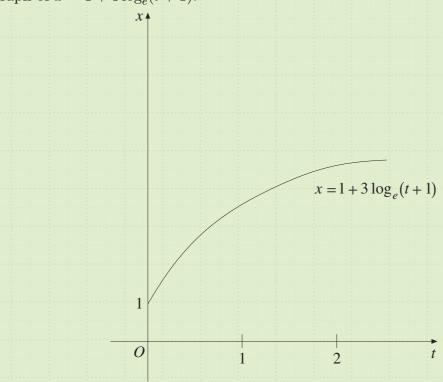
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#### 3.3.3 Optimisation/Motion



[1996 2U HSC Q9] Two particles P and Q start moving along the x axis at time t=0 and never meet. Particle P is initially at x=4 and its velocity v at time t is given by v=2t+4.

The position of particle Q is given by  $x = 1 + 3\log_e(t+1)$ . The diagram shows the graph of  $x = 1 + 3\log_e(t+1)$ .



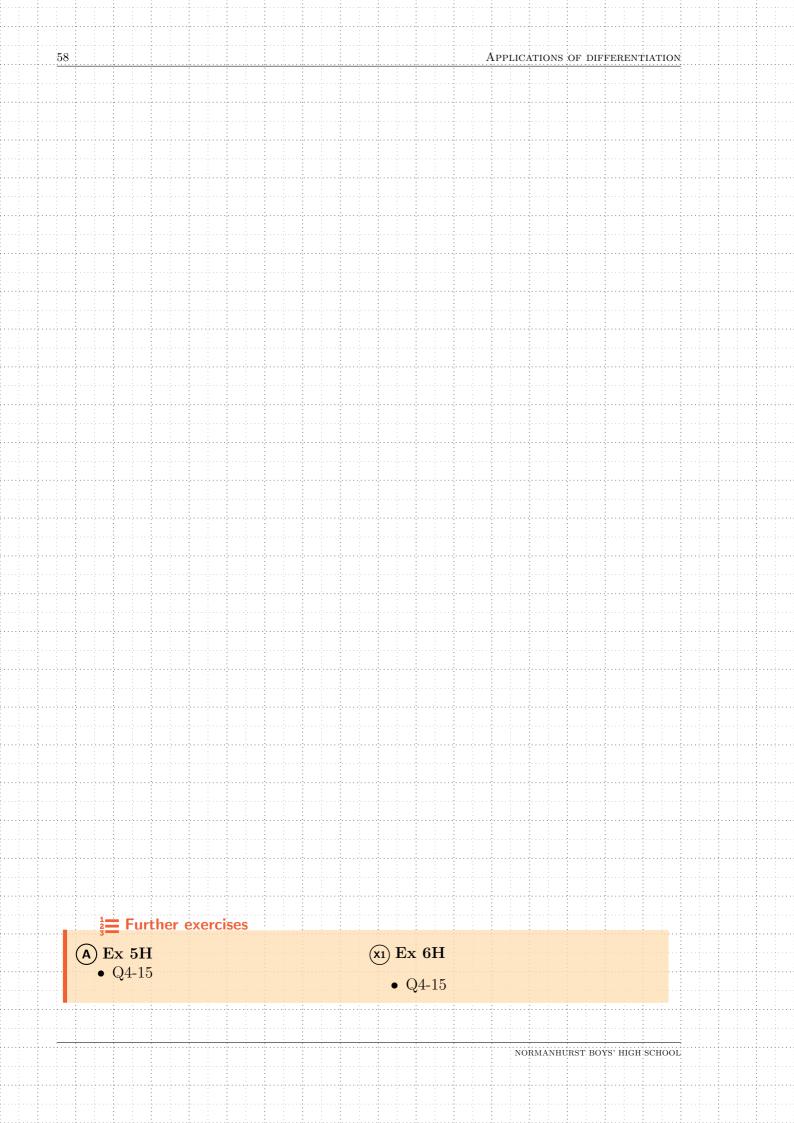
(a) Find an expression for the position of P at time t.

(b) On the same set of axes, draw the graph of the function found in part (a).

(c) P and Q are joined by an elastic string and M is the midpoint of the string. Show that the position of M at time t is given by

$$x = \frac{1}{2} \left( t^2 + 4t + 3 \log_e(t+1) + 5 \right)$$

- (d) Find the time at which the acceleration of M is zero.
- (e) Find the minimum distance between P and Q.



# Section 4

# Integration involving the logarithmic function



Various integration techniques involving the logarithmic func-

Recognising the form  $\frac{f'(x)}{f(x)}$  and finding primitives of these func-

**Variable 1** Understanding Why  $\int \frac{f'(x)}{f(x)} dx = \log_e(f(x)) + C$ 

**☑** By the end of this section am I able to:

22.13 Establish and use the formula  $\int \frac{1}{x} dx = \ln|x| + c$  and  $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$  for  $x \neq 0$ ,  $f(x) \neq 0$ 

22.14 Establish and use the formulae  $\int a^x dx = \frac{a^x}{\ln a} + c$ 

22.15 Calculate the area under a curve

22.16 Calculate areas between curves determined by any functions within the scope of this syllabus

Use the Trapezoidal rule to estimate areas under curves

## Primitives resulting in $\log_e x$

- Given  $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ , then  $\int \frac{1}{x} dx = \log_e |x| + C$ 
  - Laws/Results
  - By the chain rule,  $\frac{d}{dx}(\log_e f(x)) = \dots \frac{f'(x)}{f(x)} \dots$
- Hence,  $\int \dots \frac{f'(x)}{f(x)} \dots dx = \log_e |f(x)| + C.$

#### Standard questions

- Important note
- When presented with a rational function as the integrand, look for the form  $\dots \frac{f'(x)}{f(x)} \dots$

## Example 39

Evaluate the following:

$$1. \qquad \int \frac{dx}{2x+3}.$$

$$2. \qquad \int \frac{2}{3 - 9x} \, dx$$

$$3. \qquad \int \frac{3x}{1+6x^2} \, dx$$

$$4. \qquad \int \left(x + \frac{1}{x^2}\right)^2 dx$$

$$5. \qquad \int \frac{e^x}{1 + e^x} \, dx$$

#### Answers

$$\mathbf{1.} \ \ \frac{1}{2} \log_e(2x+3) + C \ \mathbf{2.} \ -\frac{2}{9} \log_e(3-9x) + C \ \mathbf{3.} \ \ \frac{1}{4} \log_e(1+6x^2) + C \ \mathbf{4.} \ \ \frac{1}{3} x^3 + 2 \ln x - \frac{1}{3} x^{-3} + C \ \mathbf{5.} \ \ln(1+e^x) + C \ \mathbf{6.} \ \ln(1+e^x) + C \ \mathbf{$$

#### Answers

1. 
$$\frac{1}{2}\log_e(2x+3) + C$$
 2.  $-\frac{2}{9}\log_e(3-9x) + C$  3.  $\frac{1}{4}\log_e(1+6x^2) + C$  4.  $\frac{1}{3}x^3 + 2\ln x - \frac{1}{3}x^{-3} + C$  5.  $\ln(1+e^x) + C$ 

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Find the exact value of 
$$\int_0^1 \frac{x}{x^2+3} dx$$
.

Answer: 
$$\frac{1}{2} \ln \frac{4}{3}$$

## 4.1.2 Integrands requiring manipulation

- (a) Show that  $\frac{x+7}{x-1}$  can be expressed as  $1 + \frac{8}{x-1}$ .
- (b) Hence evaluate  $\int_2^{e+1} \frac{x+7}{x-1} dx$ .

#### **Answer:** e + 7

Show that  $\frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$ . Hence or otherwise, evaluate

$$\int_2^3 \frac{1}{x^2 - 1} \, dx$$

Answer:  $\frac{1}{2} \ln \frac{3}{2}$ 

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## Example 43

- Differentiate  $\frac{\log_e x}{x}$ .
- Hence find the primitive of  $\frac{\log_e x}{x^2}$ .

**Answer:** (a) 
$$\frac{1 - \log_e x}{x^2}$$
 (b)  $-\frac{1}{x} - \frac{\log_e x}{x} + C$ 



## Example 44

Evaluate  $\frac{d}{dx}(x^2 \log_e x)$  and hence evaluate  $\int_1^e x \log_e x \, dx$ .

Answer:  $\frac{e^2+1}{4}$ 

## ½ Further exercises

- **A** Ex 5I

   Q1-4, 6 last 2 columns
- $(x_1)$  Ex 6I
  - Q2-8 last 2 columns
- Q5-18
- Q9-16

## 4.2 Applications of integration

#### 4.2.1 Simple areas

Example 45

Sketch the curve  $y = 1 - \frac{3}{x}$ . Hence or otherwise find the area bound by the curve  $y = 1 - \frac{3}{x}$ , the x axis and the line x = 1.

Answer:  $3\log_e 3 - 2$ 

## Example 46

 $[2003\ 2U\ HSC]$  - modified  $\,$  Use the Trapezoidal Rule with three function values to find an approximation for

$$\int_{2}^{6} \frac{x}{\log_{e} x} \, dx$$

Give your answer correct to 1 decimal place.

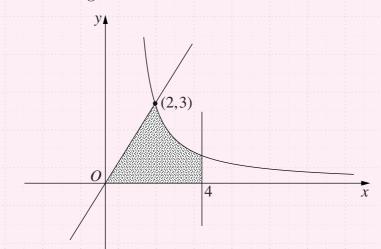
- Example 47
- (a) Sketch the curve  $y = \frac{x+1}{x}$ .
- (b) Calculate the area enclosed by the curve, the x axis and the line x = -5.

Answer:  $4 - \log_e 5$ 

**①** Timed exam practice 7 (Allow approximately 5 minutes)

[2021 Adv HSC Q24] (3 marks) The curve  $y = \frac{3}{x-1}$  intersects the line  $y = \frac{3}{2}x$  at the point (2,3).

The region bounded by the curve  $y = \frac{3}{x-1}$ , the line  $y = \frac{3}{2}x$ , the x axis and the line x = 4 is shaded in the diagram.



Find the exact area of the shaded region.

#### Marking criteria

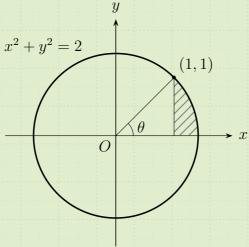
- $\checkmark$  [1] Calculates the area of the triangle, or equivalent merit
- $\checkmark$  [2] Evaluates  $\int_{2}^{4} \frac{3}{x-1} dx$ , or equivalent merit
- $\checkmark$  [3] Provides the correct solution

#### 4.2.2 Area between two curves



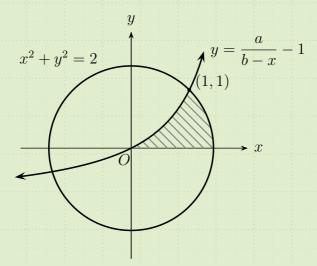
[2022 Adv HSC Q28] The graph of the circle  $x^2 + y^2 = 2$  is shown.

The interval connecting the origin, O, and the point (1,1) makes an angle  $\theta$  with the positive x axis.



(a) By considering the value of  $\theta$ , find the exact area of the shaded region, as shown on the diagram.

Part of the hyperbola  $y = \frac{a}{b-x} - 1$  which passes through the points (0,0) and (1,1) is drawn with the circle  $x^2 + y^2 = 2$  as shown.



(b) Show that a = b = 2.

3

(c) Using parts (a) and (b), find the exact area of the region bounded by the hyperbola, the positive x axis and the circle as shown on the diagram.

Answer:  $2 \ln 2 + \frac{\pi}{8} - \frac{3}{2}$ 

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## Example 49

# [2009 2U HSC Q10] Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ .

(a) Show that the graph of y = f(x) has no turning points.

(b) Find the point of inflexion of y = f(x).

(c) i. Show that  $1 - x + x^2 - \frac{1}{1+x} = \frac{x^3}{1+x}$  for  $x \neq -1$ .

ii. Let  $g(x) = \ln(1+x)$ .

Use the result in (c)i to show that  $f'(x) \ge g'(x)$  for all  $x \ge 0$ .

(d) On the same set of axes, sketch the graphs of y = f(x) and y = g(x) for  $x \ge 0$ .

(e) Show that  $\frac{d}{dx}[(1+x)\ln(1+x) - (1+x)] = \ln(1+x)$ .

(f) Find the area enclosed by the graphs of y = f(x) and y = g(x), and the straight line x = 1.

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## 4.2.3 Inequalities



Example 50

[2002 Ext 1 HSC Q6] Let n be a positive integer.

By considering the graph of  $y = \frac{1}{x}$  show that

$$\frac{1}{n+1} < \int_n^{n+1} \frac{dx}{x} < \frac{1}{n}$$

ii. Hence deduce that

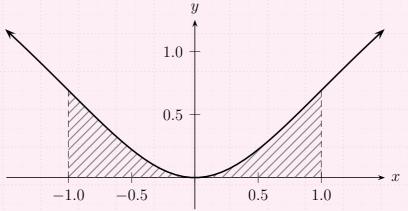
$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

3

#### 4.2.4 Trapezoidal Rule

#### **⊙** Timed exam practice 8 (Allow approximately 10 minutes)

[2024 Adv HSC Q22] (6 marks) The graph of the function  $f(x) = \ln(1 + x^2)$  is shown.



(a) Prove that f(x) is concave up for -1 < x < 1.

(b) A table of function values, correct to 4 decimal places, for some x values is provided.

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1		0	0.25	0.5	0.75	1
	x	U	0.20	0.0	0.70	1
- 1						
	$\ln{(1+x^2)}$	0	0.0606	0.2221	0.4463	0.6021
1	m(1+x)	U	0.0000	0.4451	0.4403	0.0951

Using the function values provided and the trapezoidal rule, estimate the shaded area in the diagram.

(c) Is the answer to part (b) an overestimate or underestimate? Give a reason for your answer.

Important note

⚠ The word *hence* may not appear explicitly!

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#### **4.2.5** Motion



[2000 2U HSC Q8] A particle is moving in a straight line, starting from the origin. At time t seconds the particle has a displacement of x metres from the origin and a velocity v ms<sup>-1</sup>. The displacement is given by  $x = 2t - 3\log_e(t+1)$ .

- (i) Find an expression for v
- (ii) Find the initial velocity.
- (iii) Find when the particle comes to rest.
- (iv) Find the distance travelled by the particle in the first three seconds. Answer: (i)  $\dot{x} = 2 - \frac{3}{t+1}$  (ii) -1 (iii)  $t = \frac{1}{2}$  (iv)  $4 + 3 \log_e \frac{9}{16}$  m

## Example 52

[2007 2U HSC Q5] A particle is moving on the x axis and is initially at the origin. Its velocity, v metres per second, at time t seconds is given by

$$v = \frac{2t}{16 + t^2}$$

i.	What is the initial velocity of the particle?		1
ii.	Find an expression for the acceleration of the particle.		2
iii.	Find the time when the acceleration of the particle is zero.		1
iv.	Find the position of the particle when $t = 4$ .		3

## Example 53

[2000 2U HSC Q10]  $\clubsuit$  The first snow of the season begins to fall during the night. The depth of the snow, h, increases at a constant rate through the night and the following day. At 6 am a snow plough begins to clear the road of snow. The speed, v km/h, of the snow plough is inversely proportional to the depth of snow.

(This means  $v = \frac{A}{h}$  where A is a constant)

Let x km be the distance the snow plough has cleared and let t be the time in hours from the beginning of the snowfall. Let t = T correspond to 6 am.

i. Explain carefully why, for  $t \geq T$ ,

3

$$\frac{dx}{dt} = \frac{k}{t}$$

where k is a constant.

ii. In the period from 6 am to 8 am the snow plough clears 1 km of road, but it takes a further 3.5 hours to clear the next kilometre.

At what time did it begin snowing?

Answer: 3:20 am

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## Applications of integration

- (A) Ex 5J
- Q4-18 (A) Ex 7C
  - Q8(h), Q15
- A Ex 7E
  - Q6

#### Calculus with other bases

- (A) Ex 5K
  - Q6-16

- (x1) Ex 6J
  - Q2-18
- $\bigcirc$ X1) Ex 9C
  - Q4(a)i, 11, 14
- (x1) Ex 9F
  - Q3
- (xı) Ex 6K
  - Q6-18

Watch: http://youtu.be/UFgod5tmLYY

#### **NESA** Reference Sheet – calculus based courses



**NSW Education Standards Authority** 

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced Mathematics Extension 1 Mathematics Extension 2

#### REFERENCE SHEET

#### Measurement

#### Length

$$l = \frac{\theta}{360} \times 2\pi r$$

#### A roa

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

#### Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

#### Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

#### **Functions**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For 
$$ax^3 + bx^2 + cx + d = 0$$
:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and 
$$\alpha\beta\gamma = -\frac{d}{a}$$

#### Dolotions

$$(x-h)^2 + (y-k)^2 = r^2$$

#### **Financial Mathematics**

$$A = P(1+r)^n$$

#### Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2} [2a + (n-1)d] = \frac{n}{2} (a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, r \neq 1$$

$$S = \frac{a}{1 - r}, |r| < 1$$

#### **Logarithmic and Exponential Functions**

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

#### **Trigonometric Functions**

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

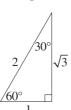
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



#### Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \cos A \neq 0$$

$$\csc A = \frac{1}{\sin A}, \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

#### Compound angles

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

If 
$$t = \tan \frac{A}{2}$$
 then  $\sin A = \frac{2t}{1+t^2}$ 

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A \cos B = \frac{1}{2} \left[ \cos(A - B) + \cos(A + B) \right]$$

$$\sin A \sin B = \frac{1}{2} \left[ \cos(A - B) - \cos(A + B) \right]$$

$$\sin A \cos B = \frac{1}{2} \left[ \sin(A+B) + \sin(A-B) \right]$$

$$\cos A \sin B = \frac{1}{2} \left[ \sin(A+B) - \sin(A-B) \right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

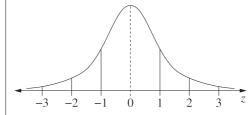
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

#### Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score less than  $Q_1$  – 1.5 × IQR or more than  $Q_3$  + 1.5 × IQR

#### **Normal distribution**



- approximately 68% of scores have z-scores between -1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$Var(X) = E[(X - \mu)^2] = E(X^2) - \mu^2$$

#### Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

#### Continuous random variables

$$P(X \le x) = \int_{a}^{x} f(x) dx$$

$$P(a < X < b) = \int_{a}^{b} f(x) dx$$

#### **Binomial distribution**

$$P(X = r) = {}^{n}C_{r}p^{r}(1-p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow P(X=x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \dots, n$$

$$E(X) = np$$

$$Var(X) = np(1-p)$$

#### **Differential Calculus**

#### **Function**

#### Derivative

$$y = f(x)^n$$

$$\frac{dy}{dx} = nf'(x) [f(x)]^{n-1}$$

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = g(u)$$
 where  $u = f(x)$   $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ 

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \sin f(x)$$

$$\frac{dy}{dx} = f'(x)\cos f(x)$$

$$y = \cos f(x)$$

$$\frac{dy}{dx} = -f'(x)\sin f(x)$$

$$y = \tan f(x)$$

$$\frac{dy}{dx} = f'(x)\sec^2 f(x)$$

$$y = e^{f(x)}$$

$$\frac{dy}{dx} = f'(x)e^{f(x)}$$

$$y = \ln f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$y = a^{f(x)}$$

$$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$$

$$y = \log_a f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{(\ln a) f(x)}$$

$$y = \sin^{-1} f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \cos^{-1} f(x)$$

$$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1 - [f(x)]^2}} \qquad \int_a^b f(x) dx$$

$$y = \tan^{-1} f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{1 + [f(x)]^2}$$

$$\approx \frac{b - a}{2n} \left\{ f(a) + f(b) + 2a \right\}$$
where  $a = x_0$  and  $b = x_n$ 

#### **Integral Calculus**

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$
where  $n \neq -1$ 

where 
$$n \neq -1$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\frac{dy}{dx} = f'(x)\cos f(x)$$

$$\frac{dy}{dx} = -f'(x)\sin f(x)$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\frac{dy}{dx} = f'(x)\sin f(x)$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\frac{dy}{dx} = f'(x)e^{f(x)}$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1} \frac{f(x)}{a} + c$$

$$\frac{dy}{dx} = (\ln a) f'(x) a^{f(x)}$$

$$\frac{dy}{dx} = \frac{f'(x)}{(\ln a) f(x)}$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1} \frac{f(x)}{a} + c$$

$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int_{a}^{b} f(x) dx$$

$$\approx \frac{b-a}{2n} \Big\{ f(a) + f(b) + 2 \Big[ f(x_1) + \dots + f(x_{n-1}) \Big] \Big\}$$

where 
$$a = x_0$$
 and  $b = x_0$ 

#### **Combinatorics**

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$${\binom{n}{r}} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^{n} = x^{n} + {\binom{n}{1}}x^{n-1}a + \dots + {\binom{n}{r}}x^{n-r}a^{r} + \dots + a^{n}$$

#### **Vectors**

$$\begin{split} \left| \stackrel{\smile}{u} \right| &= \left| x \underline{i} + y \underline{j} \right| = \sqrt{x^2 + y^2} \\ \underbrace{u \cdot y} &= \left| \stackrel{\smile}{u} \right| \left| \stackrel{\smile}{y} \right| \cos \theta = x_1 x_2 + y_1 y_2, \\ \text{where } \underbrace{u} &= x_1 \underline{i} + y_1 \underline{j} \\ \text{and } \underbrace{v} &= x_2 \underline{i} + y_2 \underline{j} \\ \underbrace{r} &= \underbrace{a} + \lambda \underline{b} \end{split}$$

#### **Complex Numbers**

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$
$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

#### **Mechanics**

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$x = a\cos(nt + \alpha) + c$$
$$x = a\sin(nt + \alpha) + c$$
$$\ddot{x} = -n^2(x - c)$$